

Barotropic-Mode Derivation Respecting Ocean-Bottom Gradients in Basin-Scale Ocean Circulation Modeling

Shinya KAKUTA^{*1}

I derived advection terms related to sea-bottom gradients from the authentic change of the order of derivative and integral operators in derivation of momentum equations of vertically integrated flow by the vertical integration of primitive equations of oceanic fluid dynamics from the sea bottom to the surface (barotropic flow). In basin-scale ocean-circulation models, these terms are customarily neglected, but may not be negligible when bottom flow is large on steep sea-bottom gradients as continental shelf breaks. This term accelerates horizontal barotropic flow regardless of the direction of the barotropic flow in the case of downward basal flow on the slope. In addition, a customarily implied assumption that horizontal flow is not correlated with its horizontal partial derivatives is not assured on shelf breaks. Distorted orthogonal coordinate system where an axis is parallel to isobath is not recommendable because of a problem of singular points.

Keywords : vertical integration, bottom gradients, basin scale, ocean circulation modeling, basal flow

海盆スケール海洋循環モデルにおいて海底勾配を 無視しない順圧モード導出

角田 晋也^{*2}

海洋流体力学のPrimitive方程式を海底から海面まで鉛直方向に積分して、鉛直方向に積分した流れ(順圧流れ)の運動方程式を導出する際に、積分と微分の作用の順番の入替を厳密に行うことにより、海底勾配に係る移流項を導出した。この項は海盆スケールの海洋循環モデルの慣例では無視されるが、陸棚斜面等、海底勾配が急峻なところでは、海底の流速が大きい場合、無視できない。斜面上の底流が流れ落ちる向きの場合、水平の順圧流れの向きに係らずこの項は順圧流れを加速する。また、海底から海面まで鉛直積分を施す際に、慣例では暗黙のうちに水平流速とその水平偏微分は無相関であると仮定するが、この仮定も陸棚斜面上では保証されない。片方の軸が等深線に平行な歪んだ直交座標系は特異点問題があるので推奨できない。

キーワード：鉛直積分、海底勾配、海盆スケール、海洋循環モデル、海底の流

*1 Ocean Observation and Research Department Japan Marine Science and Technology Center

*2 海洋科学技術センター海洋観測研究部

1. Introduction

Global ocean circulation models calculate the barotropic-mode or vertically averaged flow, which evolves slower than the baroclinic mode, with a longer time increment under the assumption of the shallow-water (long-wave) approximation, before calculation of the baroclinic mode in each layer with a much shorter time increment (e. g. Mellor and Kantha, 1989). These models smooth steep sea-bottom topography as in the Arctic Ocean to prevent numerical instability generated on the barotropic mode (e. g. Mauritzen and Häkkinen, 1997). However, excessive smoothing of bottom topography is not desirable, because topography mainly contributes to flow patterns in fact as in consideration of conservation of potential vorticity. This paper discusses where influence of bottom topography appears by the authentic change of the order of derivative and integral operators in derivation of barotropic-mode momentum equations by vertical integration of primitive momentum equations.

The primitive equations are as follows:
Continuity equation of incompressible fluid

$$\frac{\partial w}{\partial z} = -\nabla \cdot \vec{u} \quad \dots\dots\dots (1)$$

Horizontal momentum equations

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} + w \frac{\partial \vec{u}}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad \dots\dots\dots (2)$$

and Vertical momentum equations.

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} + f \vec{k} \times \vec{u} = -\frac{1}{\rho} \nabla p \quad \dots\dots\dots (3)$$

Where arrows denote 2-dimensional vectors; $\vec{k}$: unit z vector, $\vec{u}$: horizontal velocity, w : vertical velocity; p : pressure, ρ : water density, f : Coriolis parameter, g : gravity acceleration, t : time, z : vertical coordinate. In this paper, the upward direction is positive in the z axis. This paper omits viscosity terms to make equations easier to see to clarify the points.

In the following sections, the above equations are vertically integrated from the sea bottom to the surface.

2. Continuity equation

Vertical integration of Continuity equation Eq. (1) from the bottom $-H$ to the surface ζ yields the following equation:

$$w|_{z=\zeta} - w|_{z=-H} = - \int_{-H}^{\zeta} \nabla \cdot \vec{u} dz \quad \dots\dots\dots (4)$$

Here, each term can be transformed as follows; the first term in the left hand side can be rewritten as:

$$w|_{z=\zeta} = \frac{\partial \zeta}{\partial t} + \left(\vec{u} \Big|_{z=\zeta} \cdot \nabla \right) \zeta \quad \dots\dots\dots (5)$$

Assuming that bottom flow is parallel to the slope, the second term in the left hand side of Eq. (4) can be rewritten as:

$$w|_{z=-H} = -\vec{u} \Big|_{z=-H} \cdot \nabla H + \frac{1}{2} \sqrt{\frac{\nu}{f}} \vec{k} \cdot (\nabla \times \vec{u} \Big|_{z=-H}) \quad \dots\dots\dots (6)$$

Here, ν is coefficient of kinetic viscosity, and the second term in the right hand side is the Ekman pumping.

The right hand side of Eq. (4) can be rewritten as:

$$\begin{aligned} & \int_{-H}^{\zeta} \nabla \cdot \vec{u} dz \\ &= \nabla \cdot \int_{-H}^{\zeta} \vec{u} dz - \left(\vec{u} \Big|_{z=\zeta} \cdot \nabla \right) \zeta - \left(\vec{u} \Big|_{z=-H} \cdot \nabla \right) H \\ &= \nabla \cdot \vec{U} - \left(\vec{u} \Big|_{z=\zeta} \cdot \nabla \right) \zeta - \left(\vec{u} \Big|_{z=-H} \cdot \nabla \right) H \quad \dots\dots\dots (7) \end{aligned}$$

See Appendix 1 for details.

Eqs. (4) to (7) yield the following equation:

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot \vec{U} + \frac{1}{2} \sqrt{\frac{\nu}{f}} \vec{k} \cdot (\nabla \times \vec{u} \Big|_{z=-H}) \quad \dots\dots\dots (8)$$

Customarily, the second term of the right hand side is neglected, because conservation of water volume is not guaranteed with large bottom velocity.

For future use, the following notations are defined;

$$\vec{U} \equiv \int_{-H}^{\zeta} \vec{u} dz \quad \dots\dots\dots (9)$$

$$\vec{\bar{u}} \equiv \frac{\vec{U}}{\zeta + H} \quad \dots\dots\dots (10)$$

$$\vec{u} \equiv \vec{u} - \vec{\bar{u}} \quad \dots\dots\dots (11)$$

Here, the upper bar denotes vertical mean and the prime denotes difference from the mean.

Integration of continuity Equation (1) from the bottom to the surface can be rewritten by vertical mean of horizontal flow as:

$$\frac{\partial \bar{\zeta}}{\partial t} = -\nabla \cdot \{(\bar{\zeta} + H)\bar{\vec{u}}\} \\ = -(\bar{\zeta} + H)\nabla \cdot \bar{\vec{u}} - \bar{\vec{u}} \cdot \nabla \bar{\zeta} - \bar{\vec{u}} \cdot \nabla H \quad \dots\dots\dots (12)$$

Here, the Ekman pumping is neglected. The last two terms are effects of slopes of the surface and the bottom. On the shelf break, the former is negligible, because the gradients of the bottom are 1000 times larger than those of the surface.

In the case that wave length is over 20 times than bottom depth, the shallow-water wave (long wave) approximation that horizontal velocity is vertically uniform or independent of water depth is adopted. In the shallow-water wave, the vertical component of velocity is much smaller than the horizontal ones. But, in the case of steep bottom gradients, the vertical component is not negligible, because water flows on the slope. Thus, it is not in the hydrostatic balance. The vertical component of velocity is discussed in a later section.

As a criterion of plausibility of the shallow-water wave (long wave) approximation, the above equation is examined. For simplicity, supposing that the surface tilts are negligible and that the direction of the maximum gradient of the slope is parallel to the x axis, the above equation can be rewritten as:

$$\frac{\partial \bar{\zeta}}{\partial t} = -H \frac{\partial \bar{u}}{\partial t} - \bar{u} \frac{\partial H}{\partial x} \quad \dots\dots\dots (13)$$

The second term in the right hand side is an effect of the bottom gradient. Thus, if the second term is much smaller than the first term, or if

$$H \left| \frac{\partial \bar{u}}{\partial x} \right| \gg \left| \bar{u} \right| \left| \frac{\partial H}{\partial x} \right| \quad \dots\dots\dots (14)$$

then, the effect of the even steep bottom gradient is negligible. The above equation is rewritten as follows;

$$1 \gg \left| \frac{\frac{\partial}{\partial x} \ln(H)}{\frac{\partial}{\partial x} \ln|\bar{u}|} \right| \quad \dots\dots\dots (15)$$

Using the above criterion, numerical calculation results of the shallow-water wave (long wave) models can be verified. On shelf breaks, the distance between a 200 m deep point and a 2000 m deep point is, for example, 100 km. That is that bottom depth increases 10 times in 100 km. To satisfy the above equation, vertical mean of horizontal flow should decrease over 100 times in 100 km. Such distribution of flow can be realized, only if the shelf break dam the flow up. Namely, the shallow-water wave (long wave) models can be suitable, only if vertical mean of horizontal velocity is almost 0 on the shelf break in the direction of the maximum gradient of the slope. It is well-known that the steady flow, which is not density-stratified, in nearly rigid rotation is parallel to the isobath. However, unsteady flow is not always parallel to isobath.

3. Horizontal momentum equations

Integration of horizontal momentum Equation (3) yields the following equation:

$$\int_{-H}^{\bar{\zeta}} \frac{\partial \vec{u}}{\partial t} dz + \int_{-H}^{\bar{\zeta}} (\vec{u} \cdot \nabla) \vec{u} dz + f \vec{k} \times \int_{-H}^{\bar{\zeta}} \vec{u} dz \\ = -\frac{1}{\rho} \int_{-H}^{\bar{\zeta}} \nabla p dz \quad \dots\dots\dots (16)$$

3.1. Euler's derivative term

The first term of in the left hand side of the above equation can be transformed as follows:

$$\int_{-H}^{\bar{\zeta}} \frac{\partial \vec{u}}{\partial t} dz \\ = \frac{\partial}{\partial t} \int_{-H}^{\bar{\zeta}} \vec{u} dz - \left. \vec{u} \right|_{z=\bar{\zeta}} \frac{\partial \bar{\zeta}}{\partial t} \\ = \frac{\partial}{\partial t} \vec{U} - \left. \vec{u} \right|_{z=\bar{\zeta}} \frac{\partial \bar{\zeta}}{\partial t} \quad \dots\dots\dots (17)$$

See Appendix 1 for details. The last term is customarily neglected. However, this paper retains this term, where value of the Euler's differential of the surface height is given by Eq. (8).

3.2. Advection terms

Following the custom, under an implied assumption that $\vec{u}$ and $\nabla \vec{u}$ are not correlated, namely, that differences from each vertical mean are orthogonal, the second term (advection term) in the left hand side of the vertically integrated horizontal momentum equation (16) can be transformed as follows:

$$\begin{aligned}
& \int_{-H}^{\zeta} (\vec{u} \cdot \nabla) \vec{u} dz \\
&= \frac{1}{\zeta + H} \left(\int_{-H}^{\zeta} \vec{u} dz \cdot \int_{-H}^{\zeta} \nabla \right) \vec{u} dz \\
&= \frac{1}{\zeta + H} \left(\vec{U} \cdot \int_{-H}^{\zeta} \nabla \right) \vec{u} dz \\
&= \frac{1}{\zeta + H} \left\{ (\vec{U} \cdot \nabla) \int_{-H}^{\zeta} \vec{u} dz - (\vec{U} \cdot \nabla \zeta) \vec{u} \Big|_{z=\zeta} - (\vec{U} \cdot \nabla H) \vec{u} \Big|_{z=-H} \right\} \\
&= \frac{1}{\zeta + H} \left\{ (\vec{U} \cdot \nabla) \vec{U} - (\vec{U} \cdot \nabla \zeta) \vec{u} \Big|_{z=\zeta} - (\vec{U} \cdot \nabla H) \vec{u} \Big|_{z=-H} \right\} \\
&\dots\dots\dots (18)
\end{aligned}$$

See Appendix 2 for details. The last term is an effect of bottom gradients and not omitted in this paper although also customarily neglected. On shelf breaks, ∇H is 1000 times larger than $\nabla \zeta$. On the other hand, surface horizontal velocity is much larger than bottom one. Thus, the last two terms are comparable. However, the effect of the bottom gradients is not negligible when surface horizontal velocity is under 100 times larger than bottom one. From the sign, it is clear that bottom gradients accelerate the barotropic-mode flow in the case of downward basal flow on the slope regardless of the direction of the barotropic flow. Such a situation is expected to occur when the surface water is rapidly cooled down in the early winter in the Arctic marginal ice zone on the Arctic continental shelf to generate dense brine water flowing down under ice formation. Neglect of the bottom gradient term causes underestimate of the barotropic flow over the shelf resulting in overestimate of rise of the surface upstream and of sink downstream by the shelf.

Here, the first and second terms are evaluated as follows:

$$|\nabla \vec{U}| \approx |\nabla \cdot \vec{U}| = \left| \frac{\partial \zeta}{\partial t} \right| \dots\dots\dots (19)$$

$$|\nabla \zeta| \left| \vec{u} \Big|_{z=\zeta} \right| \approx \left| \left(\vec{u} \Big|_{z=\zeta} \cdot \nabla \right) \zeta \right| \dots\dots\dots (20)$$

The last terms of the above equations are Euler's differential and a advection term. The smaller amplitude than wave length, the smaller the advection term than the Euler's differential one. Thus, in the right hand side of Eq. (18), the second term is expected to be smaller than the first one.

3.3. Vertically integrated horizontal momentum equations

After all, the vertically integrated horizontal momentum equations are as follows:

$$\begin{aligned}
& \frac{\partial \vec{U}}{\partial t} + \frac{1}{\zeta + H} (\vec{U} \cdot \nabla) \vec{U} + f \vec{k} \times \vec{U} \\
& - \vec{u} \Big|_{z=\zeta} \frac{\partial \zeta}{\partial t} - \frac{1}{\zeta + H} \left\{ (\vec{U} \cdot \nabla \zeta) - \vec{u} \Big|_{z=\zeta} + (\vec{U} \cdot \nabla H) - \vec{u} \Big|_{z=-H} \right\} \\
& = \int_{-H}^{\zeta} \frac{1}{\rho} \nabla p dz \dots\dots\dots (21)
\end{aligned}$$

With amplitude much smaller than wave length, advection terms are negligible. Thus, the advection term in the vertically integrated horizontal momentum equations is also expected to be small except for steep bottom gradients. Although this last term in the left hand side is neglected in customary derivation of the barotropic mode, it is not negligible on steep bottom gradients. The next section discusses how to determine value of pressure in the right hand side.

4. Vertical momentum equation

In this section, the vertical momentum Equation (3) is vertically integrated.

4.1. Procedure of numerical calculation of value of pressure

How can value of pressure in the right hand side of Eq. (21) be obtained? The value is given by vertical integration of the vertical partial derivative of pressure in the vertical momentum Equation (3).

$$\begin{aligned}
p &= -g \int_z^{\zeta} \rho dz - \int_z^{\zeta} \rho \frac{\partial w}{\partial t} dz - \int_z^{\zeta} \rho \left\{ (\vec{u} \cdot \nabla) w + w \frac{\partial w}{\partial z} \right\} dz \\
&= -g \int_z^{\zeta} \rho dz - \bar{\rho}^z \frac{\partial}{\partial t} \int_z^{\zeta} w dz + \bar{\rho}^z w \Big|_{z=\zeta} \frac{\partial \zeta}{\partial t} \\
& - \int_z^{\zeta} \rho \left\{ (\vec{u} \cdot \nabla) w + w \frac{\partial w}{\partial z} \right\} dz \dots\dots\dots (22)
\end{aligned}$$

Here, the upper bar over ρ denotes vertical mean from z to the surface. It is assumed that there is no correlation between ρ and the vertical partial derivative of vertical velocity. Euler's differential terms appear in the second and third terms in the right hand side. Value of the Euler's differential of surface height is given by Eq. (8). Thus, the second term in the right hand side of the above equation remains to be unknown. If this term is

neglected, value of pressure can be determined. Substituting the value of p into Eq. (2) and (21), u and U are forwarded with a shorter and longer time step respectively, because U evolves slowly than u . Substituting the value of new u into Eq. (1), value of new w is obtained. Using the value, the second term of the right hand side of Eq. (22) can be evaluated. Then, it is verified whether it is justifiable to neglect the second term.

On the bottom, there is no neighboring grid to differentiate in the forth term of the right hand side. This problem is solved without elaboration, if σ vertical coordinates are adopted, where the surface is 0 and the bottom is 1. Value of w on the bottom is given by Eq. (6).

4.2. Vertically integrated vertical velocity

The following equation is obtained integrating the vertical momentum Equation (3) from the bottom to the surface as the horizontal momentum equation.

$$\begin{aligned} & \frac{\partial W}{\partial t} - w|_{z=\zeta} \frac{\partial \zeta}{\partial t} + \int_{-H}^{\zeta} (\vec{u} \cdot \nabla) w dz \\ & + \frac{1}{2} (w|_{z=\zeta})^2 - \frac{1}{2} (w|_{z=-H})^2 \\ & = \int_{-H}^{\zeta} \frac{1}{\rho} \frac{\partial p}{\partial z} dz - g(\zeta + H) \end{aligned} \quad \dots\dots\dots (23)$$

Here, W is defined as:

$$W \equiv \int_{-H}^{\zeta} w dz \quad \dots\dots\dots (24)$$

Assuming that u and ∇w are not correlated, Eq. (23) can be transformed as follows;

$$\begin{aligned} & \frac{\partial W}{\partial t} - w|_{z=\zeta} \frac{\partial \zeta}{\partial t} + \frac{1}{\zeta + H} \left\{ (\vec{U} \cdot \nabla) W \right. \\ & \left. - (\vec{U} \cdot \nabla \zeta) w|_{z=\zeta} - (\vec{U} \cdot \nabla H) w|_{z=-H} \right\} \\ & + \frac{1}{2} (w|_{z=\zeta})^2 - \frac{1}{2} (w|_{z=-H})^2 \\ & = \int_{-H}^{\zeta} \frac{1}{\rho} \frac{\partial p}{\partial z} dz - g(\zeta + H) \end{aligned} \quad \dots\dots\dots (25)$$

Here, influence of sea-bottom gradients appears in the advection term as in Eq (21). Assuming that basal flow is parallel to the slope as in Section 2, the term can be rewritten by Eq. (6).

$$\begin{aligned} & -\frac{1}{\zeta + H} (\vec{U} \cdot \nabla H) w|_{z=-H} \\ & = (\vec{u} \cdot \nabla H) (\vec{u}|_{z=-H} \cdot \nabla H) \end{aligned} \quad \dots\dots\dots (26)$$

Here, the Ekman pumping term is omitted. On shelf breaks, value of Eq. (26) is about 10000 times smaller than square of horizontal velocity, because ∇H is about 0.01 there. In such consideration as in Section 3.2, the advection terms in Eq. (25) are, roughly speaking, in the same magnitude. However, the influence of sea-bottom gradients is dominant, when basal flow is large. On the other hand, the terms in the right hand side almost cancels each other nearly in hydrostatic balance. Otherwise, the larger value of the right hand side, the more out of hydrostatic balance it is.

Eq. (8) gives value to the Euler's differential of surface height in the second term in the left hand side of Eq. (23). Thus, the first term in the left hand side can be evaluated by Eq. (23) because the only unknown. If value of the term were large, it would be attributed to irrelevance of the assumption that there is no correlation between water density and Euler's differential of vertical velocity under the approximation that the second term in the right hand side of Eq. (22) is 0.

5. Discussion

5.1. Assumption of no correlation

The following is assumed on vertical integration of advection terms in Sections 3.2. and 4.2.;

$$\int_{-H}^{\zeta} (\vec{u}' \cdot \nabla) \vec{u}' = 0 \quad \dots\dots\dots (27)$$

$$\int_{-H}^{\zeta} (\vec{u}' \cdot \nabla) w' dz = 0 \quad \dots\dots\dots (28)$$

Here, the prime refers to difference from vertical mean from the bottom to the surface.

The following is assumed in Eq. (22).

$$\int_{\zeta}^{\zeta} \left(\rho - \frac{1}{\zeta - z} \int_{\zeta}^{\zeta} \rho dz \right) \left(\frac{\partial w}{\partial t} - \frac{1}{\zeta - z} \int_{\zeta}^{\zeta} \frac{\partial w}{\partial t} dz \right) dz = 0 \quad \dots\dots\dots (29)$$

The above assumptions are verified with value of u and w obtained from short-time-step integration. If they are irrelevant, these terms should not be neglected. Intuition expects that, on a shelf break, u largely varies between depths over and under a continental shelf and thus that, the larger u , the larger ∇u . But, Eq. (27) is customarily implied. It remains to be studied whether it is relevant to approximate a higher-order moment not as 0 but as a simple function of lower-order as in turbulent

closure models. Here, the first-order moment refers to vertical mean from the bottom to the surface.

5.2. Horizontal coordinate system

In momentum Equations (22) and (25) integrated vertically from the bottom to the surface, a term containing bottom gradients appears. However, in the location where isobath is parallel to x or y axis, a bottom gradient disappears in the axis parallel to isobath. Thus, coordinate transform from Cartesian x-y to isobath-orthogonal to it seems to have advantage. However, such a distorted orthogonal coordinate system has singular points at local minimum of bottom depth because isobath coordinates regress there. There, horizontal partial derivatives cannot be defined except divergence and rotation operators. A similar singular-point problem occurs at the North pole in the spherical polar coordinate system of longitudes and latitudes. To avoid such singular points, for example, Tomita et al (2001) proposed a model using a modified icosahedral geodesic grid.

6. Conclusions

I derived advection terms related to sea-bottom gradients from the authentic change of the order of derivative and integral operators in derivation of momentum equations of vertically integrated flow by the vertical integration of primitive equations of oceanic fluid dynamics from the sea bottom to the surface (barotropic flow). In basin-scale ocean- circulation models, these terms are customarily neglected, but may not be negligible when bottom flow is large on steep sea-bottom gradients as continental shelf breaks. Sea-bottom gradients accelerate the barotropic-mode flow in the case of downward basal flow on the slope regardless of the direction of the barotropic flow.

In addition, a customarily implied assumption that horizontal flow is not correlated with its horizontal partial derivatives is not assured on shelf breaks. This assumption is used in derivation of momentum equations of barotropic flow from primitive ones.

Distorted orthogonal coordinate system where an axis is parallel to isobath is not recommendable because of a problem of singular points.

Appendix 1

The first term of vertically integrated horizontal momentum Equation (16) can be transformed as follows;

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{-H}^{\zeta} \vec{u} dz \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \int_{-(H+\Delta H)}^{\zeta+\Delta \zeta} (\vec{u} + \Delta \vec{u}) dz - \int_{-H}^{\zeta} \vec{u} dz \right\} \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \int_{-(H+\Delta H)}^{-H} (\vec{u} + \Delta \vec{u}) dz + \int_{-H}^{\zeta} (\vec{u} + \Delta \vec{u}) dz \right. \\ & \quad \left. + \int_{\zeta}^{\zeta+\Delta \zeta} (\vec{u} + \Delta \vec{u}) dz - \int_{-H}^{\zeta} \vec{u} dz \right\} \\ &= \lim_{\Delta t \rightarrow 0} \left\{ \frac{\Delta H}{\Delta t} (\vec{u} + \Delta \vec{u}) + \int_{-H}^{\zeta} \frac{\Delta \vec{u}}{\Delta t} dz + \frac{\Delta \zeta}{\Delta t} (\vec{u} + \Delta \vec{u}) \right\} \\ & \frac{\partial H}{\partial t} \vec{u} + \int_{-H}^{\zeta} \frac{\partial \vec{u}}{\partial t} dz + \frac{\partial \zeta}{\partial t} \vec{u} \dots\dots\dots (A1.1) \end{aligned}$$

Here Δ denotes increment in Δt . Euler's differential of bottom depth H is 0 because H is constant in time. Note that H is always positive while z coordinate of the bottom $-H$ is always negative. The positive direction of z axis is upward.

Eq. (7) is also derived in a similar way.

Appendix 2

Following the custom, under assumption that there is no correlation between u and ∇u , namely, differences from their vertical means are orthogonal;

$$\int_{-H}^{\zeta} \vec{u}' \cdot (\nabla \vec{u})' dz = 0 \dots\dots\dots (A2.1)$$

Here, a prime refers to difference from the vertical mean.

$$\begin{aligned} & \int_{-H}^{\zeta} (\vec{u} \cdot \nabla) \vec{u} dz \\ &= \int_{-H}^{\zeta} (\vec{u} + \vec{u}') \cdot \left\{ \overline{\nabla \vec{u}} + (\nabla \vec{u})' \right\} dz \\ &= (\zeta + H) \vec{u} \cdot \overline{\nabla \vec{u}} + \int_{-H}^{\zeta} \vec{u}' \cdot (\nabla \vec{u})' dz \\ &= \frac{1}{(\zeta + H)} \int_{-H}^{\zeta} \vec{u} dz \cdot \int_{-H}^{\zeta} \nabla \vec{u} dz + \int_{-H}^{\zeta} \vec{u}' \cdot (\nabla \vec{u})' dz \\ & \dots\dots\dots (A2.2) \end{aligned}$$

This assumption seems to be customarily overlooked rather than adopted.

References

- Mellor G. L. and Kantha, L., An ice-ocean coupled model, Journal of Geophysical Research, 94, pp. 10937-10954 (1989)
- Mauritzen C. and Häkkinen S., Influence of sea ice on the thermohaline circulation in the Arctic-North Atlantic Ocean, Geophysical Research Letters, 24, pp. 3257-3260 (1997)
- Tomita H., Tsugawa M., Satoh M., and Goto K., Shallow water model on a modified icosahedral geodesic grid by using spring dynamics, Journal of Computational Physics, 174, pp. 579-613 (2001)

(Manuscript received 14 November 2002)